Psychology 594
Multivariate Analysis

Solve all the problems first by hand; redo the numerical analyses with MATLAB to check your results. Note: you don't need to do any symbolic work in MATLAB; only reproduce the numerical results.

Homework I

Problem 1:
Let $\mathbf{x}' = [6, 2, 1]$ and $\mathbf{y}' = [-1, 3, 1]$.

(a) Graph the two vectors.

(b) Find (i) the length of $\mathbf{x}$, (ii) the angle between $\mathbf{x}$ and $\mathbf{y}$, and (iii) the projection of $\mathbf{y}$ on $\mathbf{x}$.

(c) Because $\bar{x} = 3$ and $\bar{y} = 1$, graph the mean centered vectors, $[6-3, 2-3, 1-3] = [3, -1, -2]$ and $[-1-1, 3-1, 1-1] = [-2, 2, 0]$. Calculate the correlation between the three observation pairs. Find the cosine of the angle between the two mean-corrected vectors, and comment on the relation to the correlation.

Problem 2:
Given the matrices

$$\mathbf{A} = \begin{bmatrix} 1 & 4 \\ 2 & 6 \\ 3 & 8 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix}, \qquad \mathbf{C} = \begin{bmatrix} 5 \\ -4 \\ 2 \end{bmatrix}$$

perform the indicated multiplications:

(a) $5\mathbf{A}$

(b) $\mathbf{AB}$

(c) $\mathbf{B}'\mathbf{A}'$

(d) $\mathbf{C}'\mathbf{A}$

(e) Is $\mathbf{BA}$ defined? If so, calculate it.

Problem 3:

Verify the following properties of the transpose and inverse when

$$\mathbf{A} = \begin{bmatrix} 2 & 4 \\ 3 & 1 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix}$$

(a) $(\mathbf{A}')' = \mathbf{A}$

(b) $(\mathbf{B}')^{-1} = (\mathbf{B}^{-1})'$

(c) $(\mathbf{AB})' = \mathbf{B}'\mathbf{A}'$

(d) $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$

Problem 4:
Verify that

$$\mathbf{Q} = \frac{1}{3}\begin{bmatrix} 2 & -2 & 1 \\ 1 & 2 & 2 \\ 2 & 1 & -2 \end{bmatrix}$$

is an orthogonal matrix; calculate $\mathbf{QQ}'$, $\mathbf{Q}'\mathbf{Q}$, and $\mathbf{Q}^{-1}$.

Homework II

Problem 1:

Let

$$\mathbf{A} = \begin{bmatrix} 9 & -3 \\ -3 & 1 \end{bmatrix}$$

(a) Is $\mathbf{A}$ symmetric?

(b) Is $\mathbf{A}$ positive definite?

Problem 2:

Let

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix}$$

a) Determine the eigenvalues and associated eigenvectors of $\mathbf{A}$. Find the spectral decomposition of $\mathbf{A}$.

b) Find $\mathbf{A}^{-1}$.

c) Compute the eigenvalues and eigenvectors of $\mathbf{A}^{-1}$, and write out the spectral decomposition of $\mathbf{A}^{-1}$. Compare this spectral decomposition with that for $\mathbf{A}$.

Problem 3:

A quadratic form $\mathbf{x}'\mathbf{A}\mathbf{x}$ is said to be positive definite if the matrix $\mathbf{A}$ is positive definite. Is the quadratic form, $4x_1^2 + 4x_2^2 - 6x_1x_2$, positive definite?

Problem 4:

Determine the square root matrix $\mathbf{A}^{1/2}$ using the matrix $\mathbf{A}$ from the previous problem 3. Also, determine $\mathbf{A}^{-1/2}$, and show that $\mathbf{A}^{1/2}\mathbf{A}^{-1/2} = \mathbf{A}^{-1/2}\mathbf{A}^{1/2} = \mathbf{I}$.

Problem 5:

(a) Consider an arbitrary $n \times p$ matrix $\mathbf{A}$. Then $\mathbf{A}'\mathbf{A}$ is a symmetric $p \times p$ matrix. Show that $\mathbf{A}'\mathbf{A}$ is necessarily positive semi-definite. (Hint: set $\mathbf{y} = \mathbf{A}\mathbf{x}$ so that $\mathbf{y}'\mathbf{y} = \mathbf{x}'\mathbf{A}'\mathbf{A}\mathbf{x}$)

(b) Using the matrix

$$\mathbf{A} = \begin{bmatrix} 4 & 8 & 8 \\ 3 & 6 & -9 \end{bmatrix}$$

(1) Calculate $\mathbf{A}\mathbf{A}'$ and obtain its eigenvalues and eigenvectors.

(2) Calculate $\mathbf{A}'\mathbf{A}$ and obtain its eigenvalues and eigenvectors. Check that the nonzero eigenvalues are the same as those in (1).

(3) Obtain the singular-value decomposition of $\mathbf{A}$.

Homework III

Problem 1:

Let $\mathbf{X}$ have covariance matrix

$$\boldsymbol{\Sigma} = \begin{bmatrix} 25 & 0 & 2 \\ 0 & 1 & 1 \\ 2 & 1 & 9 \end{bmatrix}$$

(a) Determine $\boldsymbol{\rho}$ (the correlation matrix) and $\mathbf{V}^{1/2}$ (a diagonal matrix containing the standard deviations along the main diagonal).

(b) Multiply your matrices to check the relation $\mathbf{V}^{1/2}\boldsymbol{\rho}\mathbf{V}^{1/2} = \boldsymbol{\Sigma}$.

(c) Find ρ_{13}.

(d) Find the correlation between X_1 and $\frac{1}{4}X_2 + \frac{1}{2}X_3$.

Problem 2:

(a) Derive expressions for the mean and variances of the following linear combinations in terms of the means and covariances of the random variable X_1, X_2, and X_3.

1) $X_1 - 2X_2$
2) $-X_1 + 3X_2$
3) $X_1 + X_2 + X_3$
4) $X_1 + 2X_2 - X_3$
5) $3X_1 - 4X_2$, when X_1 and X_2 are independent random variables.

(b) The random vector $\mathbf{X}' = [X_1, X_2, X_3, X_4]$ has mean vector $[4, 3, 2, 1]$ and variance-covariance matrix

$$\begin{bmatrix} 3 & 0 & 2 & 2 \\ 0 & 1 & 1 & 0 \\ 2 & 1 & 9 & -2 \\ 2 & 0 & -2 & 4 \end{bmatrix}$$

Let $\mathbf{X}^{(1)} = [X_1, X_2]'$ and $\mathbf{X}^{(2)} = [X_3, X_4]'$, and let $\mathbf{A} = [1 \;\; 2]$ and

$$\mathbf{B} = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$$

Find: $E(\mathbf{X}^{(1)})$; $E(\mathbf{AX}^{(1)})$; $E(\mathbf{X}^{(2)})$; $E(\mathbf{BX}^{(2)})$; $Cov(\mathbf{X}^{(1)})$; $Cov(\mathbf{AX}^{(1)})$; $Cov(\mathbf{X}^{(2)})$; $Cov(\mathbf{BX}^{(2)})$; $Cov(\mathbf{X}^{(1)}, \mathbf{X}^{(2)})$; $Cov(\mathbf{AX}^{(1)}, \mathbf{BX}^{(2)})$.

For

$$\mathbf{A} = \begin{bmatrix} 9 & -2 \\ -2 & 6 \end{bmatrix}$$

find the maximum value of $\mathbf{x}'\mathbf{A}\mathbf{x}$ for $\mathbf{x}'\mathbf{x} = 1$.

Problem 3:

Give your own numerically specified 4×3 matrix, say, $\mathbf{A}$, and do the MATLAB operations of `rank`, `corrcoef`, `cov`, `mean`, `median`, `std`, and `sum` on $\mathbf{A}$, and `inv`, `trace`, `det`, `eig`, `svd`, `poly`, and `sqrtm` on $\mathbf{A}'\mathbf{A}$.