5

MDS and Facet Theory

Regional interpretations of MDS solutions are very general and particularly successful approaches for linking MDS configurations and substantive knowledge about the represented objects. Facet theory (FT) provides a systematic framework for regional interpretations. FT structures a domain of interest by partitioning it into types. The typology is generated by coding the objects of interest on some facets of their content. The logic is similar to stratifying a sample of persons or constructing stimuli in a factorial design. What is then tested by MDS is whether the distinctions made on the conceptual (design) side are mirrored in the MDS representation of the objects' similarity coefficients such that different types of objects fall into different regions of the MDS space.

5.1 Facets and Regions in MDS Space

Interpreting an MDS solution means linking geometric properties of the configuration to substantive features of the represented objects. A very general approach is to interpret regions of an MDS space. Regional interpretations are put into a systematic framework in facet theory (Guttman, 1959, 1991; Borg & Shye, 1995).

Elements of Facet Theory

The central notion of facet theory (FT) is that of a *facet*. A facet is a scheme used to classify the elements of a domain of interest into types. The facet "gender", for example, classifies persons into males and females. Similarly, the facet "behavior modality" classifies attitudinal behavior into emotional, cognitive, and actional behavior. Using several facets at the same time partitions a domain of interest into multifaceted types. Consider the tasks contained in an intelligence test, for example. In FT, such tasks are *intelligence items*, defined as questions that ask about an individual's behavior and assess it on a scale from "very right" to "very wrong" according to an objective rule (Guttman, 1965). A particular case of intelligence items are the tests in paper-and-pencil intelligence test batteries. Such tests require the testee to find verbal analogies, solve arithmetic problems, and identify patterns that complete series of figures, for example. Hence, they can be classified by the facet "language of presentation" into numerical, verbal, and geometrical ones. At the same time, such tests relate to different abilities, which gives rise to a second facet, "required mental operation". It classifies tests into those where the testee has to infer, apply, or learn a rule, respectively (Guttman & Levy, 1991). In combination, these two facets distinguish nine types of intelligence: numerical tests requiring the testee to infer a rule, numerical tests requiring the testee to apply a rule, ..., geometrical tests requiring the testee to learn a rule.

In FT, facets are typically not just listed but rather expressed in the framework of a *mapping sentence*. It shows the roles the facets play relative to each other and relative to what is being observed, that is, the *range* of the items. An example is the following.

$$\text{Person } \{p\} \text{ performs on a task presented in}$$

$$\overset{\underline{\text{language}}}{\left\{\begin{array}{c}\text{verbal}\\ \text{numerical}\\ \text{geometrical}\end{array}\right\}} \text{ language and requiring } \overset{\underline{\text{requirement}}}{\left\{\begin{array}{c}\text{learning}\\ \text{applying}\\ \text{inferring}\end{array}\right\}} \text{ an}$$

$$\text{objective rule} \rightarrow \overset{\underline{\text{range}}}{\left\{\begin{array}{c}\text{very right}\\ \text{to}\\ \text{very wrong}\end{array}\right\}} \text{ according to that rule.}$$

The terms enclosed in braces denote the facets.[1] The set of persons, p, is not stratified further in this example, whereas the questions are structured

[1] Instead of braces, one often uses vertical arrays of parentheses. Braces, however, correspond to the usual mathematical notation for listing the elements of a set. Formally, a facet is a set or, more precisely, a component set of a Cartesian product.

by the two facets from above, "requirement" and "language". The *range* of the mapping sentence is the scale on the right-hand side of the arrow. The arrow symbolizes an observational mapping of every person in p crossed with every (doubly coded) test into the range (data). Each such mapping specifies the response of a given person to a particular *type* of question. For each question type, there are generally thousands of concrete items.

Facets are invented for a particular *purpose*, that is, for systematically breaking up a domain of interest into subcategories or types in order to conceptually structure this domain. Take plants, for example. Botanists, painters, children, perfume makers, and the like, all invented category systems that allow them to order plants in some way that is meaningful for them. Good classification systems allow the user to unambiguously place each and every object into one and only one category. But good classification systems also serve a particular purpose beyond providing conceptual control: the different types distinguished by the classification system should, in one way or another, "behave" differently in real life. Whether this is true can be tested empirically and, hence, implies a hypothesis.

Facet Theory and Regions in MDS Spaces

A traditional specification of the hypothesis of empirical usefulness of a facet is that it should explain the data in some way. One way of testing this is to check whether the *distinctions* made by the facets are mirrored, facet by facet, in corresponding *differences* of the data. For example, tests that require the testee to infer, apply, or learn a rule, should lead to different responses of the testee. One particular specification of what is meant by "different" is that inferential tests are most difficult, in general, and learning tests are least difficult, with application tests in between. Another form of hypothesis is that different item types fall into different regions of an MDS representation of the item intercorrelations.

A regional hypothesis thus links content facets to regions of the empirical MDS space. The hypothesis is that the MDS space can be partitioned such that each region represents a different facet element.[2] That is, all points within a particular region should be associated with the same facet element, and points in different regions should be associated with different facet elements.

Consider an example. Table 5.1 shows the intercorrelations of eight intelligence test items, together with *structuples*, that is, codings of the items on the facets "language" and "requirement" discussed above. Item 1 in Ta-

[2]In a plane, a region is defined as a connected set of points such as the inside of a rectangle or a circle. More generally, a set of points is connected if each pair of its points can be joined by a curve all of whose points are in the set. Partitioning a set of points into regions means to split the set into classes such that each point belongs to exactly one class.

TABLE 5.1. Intercorrelations of eight intelligence tests, together with content codings on the facets “language” = {N = numerical, G = geometrical} and “requirement” = {A = application, I = inference} (Guttman, 1965).

Language	Requirement	Test	1	2	3	4	5	6	7	8
N	A	1	1.00	.67	.40	.19	.12	.25	.26	.39
N	A	2	.67	1.00	.50	.26	.20	.28	.26	.38
N	I	3	.40	.50	1.00	.52	.39	.31	.18	.24
G	I	4	.19	.26	.52	1.00	.55	.49	.25	.22
G	I	5	.12	.20	.39	.55	1.00	.46	.29	.14
G	A	6	.25	.28	.31	.49	.46	1.00	.42	.38
G	A	7	.26	.26	.18	.25	.29	.42	1.00	.40
G	A	8	.39	.38	.24	.22	.14	.38	.40	1.00

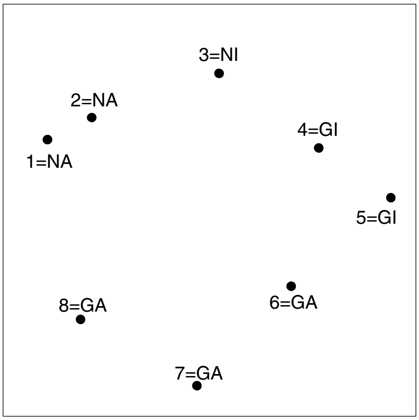

FIGURE 5.1. 2D MDS of correlations in Table 5.1.

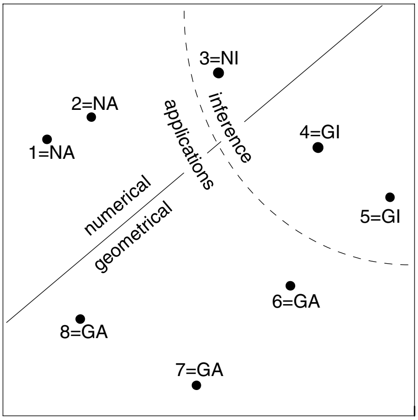

FIGURE 5.2. MDS space with four regions resulting from G- vs. N-, and A- vs. I-distinctions, respectively.

ble 5.1 is coded as numeric (on the facet “language”) and as application (on the facet “requirement”), whereas item 5 is geometrical and inference. Rather than looking at these correlations directly, we represent them in a 2D MDS space (Figure 5.1). This can be done with the low Stress of .015.

Figure 5.2 demonstrates that the MDS configuration can indeed be cut such that each partitioning line splits it into two regions containing only points of one type: points of the N-type lie above the solid line, and points of the G-type below that line. The dashed line separates I-type points from A-type points. One notes in Figure 5.2 that there is considerable leeway in choosing the partitioning lines. Why, then, was a curved line chosen for separating I-type points from A-type points? The reason is that this line yields a structure that looks like a slice from the *universe* of all possible item types discriminated by the given two facets. If items of all nine types

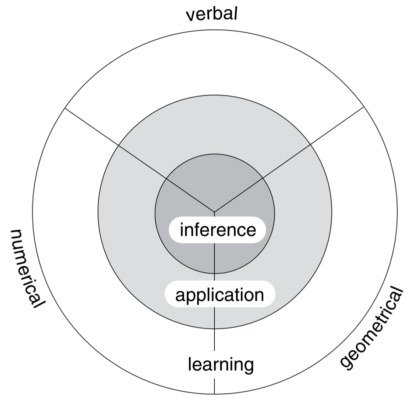

FIGURE 5.3. Schematic radex of intelligence items.

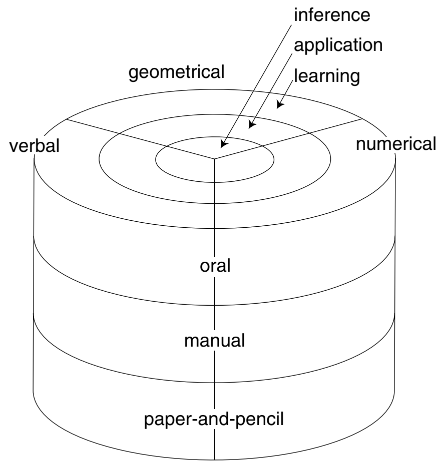

FIGURE 5.4. Cylindrex of intelligence items (after Guttman & Levy, 1991).

had been observed, one can predict that the MDS configuration would form a pattern similar to a dart board, or *radex*, shown schematically in Figure 5.3. If, in addition, one adds another facet, "communication", which distinguishes among oral, manual, and paper-and-pencil items, one obtains a 3D *cylindrex*, shown in Figure 5.4. In the cylindrex, "communication" plays the role of an axis along which the radexes for items using a fixed form of communication are stacked on top of each other.

Summarizing, we see that every facet contains additional information on the items in MDS. In a way, a facet can be seen as a design variable of the items: every item belongs to one of the categories of each and every facet. The facets are combined into a mapping sentence so that every item corresponds to one particular way of reading this mapping sentence. Some combinations of the categories may not be expressed by items, whereas other combinations may have more than one item. The facets and their categories (elements) are chosen on substantive grounds. Given a set of items classified by such facets, MDS tests whether the classification is reflected in a corresponding regionality of the representation space.

5.2 Regional Laws

The cylindrex structure has been confirmed so often for intelligence test items that now it is considered a *regional law* (Guttman & Levy, 1991). What Figure 5.2 shows, therefore, is a partial replication of the cylindrex law.

What does such a regional law mean? First of all, it reflects regularities in the data. For example, restricting oneself to items formulated in a particular language (such as paper-and-pencil tests) and, thus, to a radex as in Figure 5.3, one notes that inference items generally correlate higher among each other than application items, and learning items are least correlated. Thus, knowing that some person performs well on a given inference item allows one to predict that he or she will most likely also perform well on other inference items, whereas good performance on a given learning item says little about the performance on other learning items. One can improve the predictions, however, if one constrains them to learning tasks that use a particular language of presentation such as numerical tasks.

One notes, moreover, that the MDS regions for inference, application, and learning are ordered. This order cannot be predicted or explained from the properties of the qualitative facet "requirement", but it reliably shows up in hundreds of replications (Guttman & Levy, 1991). Thus, it seems unavoidable to ask for an explanation for this lawfulness. Ideally, what one wants is a definitional system that allows one to *formally derive* such ordered regions from its facets.

Snow, Kyllonen, and Marshalek (1984) proposed an explanation in this direction. They report a factor analysis that suggests that items which relate to points in the center of the radex (i.e., inference tasks) are "complex" items and those represented at the periphery (such as learning tasks) are "specific" items. This repeats, to some extent, what the radex says: items whose points are closer to the origin of the radex tend to be more highly correlated with other items. Snow et al. (1984) add, however, that more complex tasks show "increased involvement of one or more centrally important components." Hence, their explanation for the inference-application-learning order seems to be that these facet elements are but discrete semantic simplifications of a smooth gradient of complexity.

One can ask the complexity question in a different way and define a task t_1 as more complex than t_2 if "it requires everything t_1 does, and more" (Guttman, 1954, p. 269). Formally, this implies an interlocking of content structuples, which is analogous to the perfect Guttman scale. Specifying such structuples requires one to identify basic content facets with a common range, where the concepts "inference", "application", and "learning" then become only global labels for comparable (hence ordered) content structuples of these underlying facets. For a fixed element of the "language" facet, such a system would allow one to predict a particular order of regions (*simplex*).

But this leads to the question of what pulls the different simplexes—one for each type of required mental operation, that is, one for items that require application, learning, or inference of an objective rule, respectively—to a common origin? To explain this empirical structure requires an additional pattern in the structuples. Formally, for the three directions of the intelligence radex, it would suffice to have an additional coding of the items in

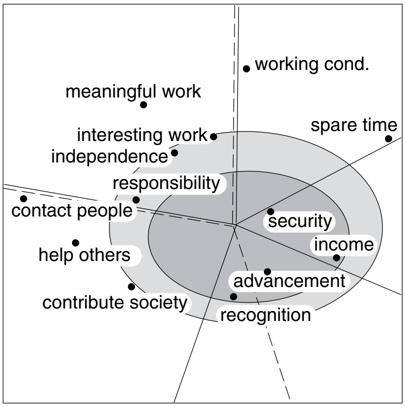

FIGURE 5.5. Radex partitionings of 13 work value items.

terms of the extent to which they require each of the three mental operations. In any case, with many points and/or differentiated facets, a simple correspondence between regions and structuples is a remarkable finding. Arbitrary assignments of structuples to the points do not, in general, lead to such lawfulness. Partitionings with relatively smooth cutting lines are generally also more reliable. Moreover, they help clarify the roles the various facets play with respect to the data. Such roles are reflected in the particular ways in which they cut the space.

5.3 Multiple Facetizations

A given object of interest can always be facetized in more than one way. Every new facet offers a new alternative. But then one may ask whether each such facetization is reflected in different statistical effects on the data side. Consider work values, for example. Work value items ask the respondent to assess the importance of different outcomes of his or her work. An example is the questionnaire item: "How important is it to you personally to make a lot of money?" with the range "very important . . . not important at all." Conceptually, two different kind of facets have been proposed for organizing such items: one facet distinguishes the work outcomes in terms of the need they satisfy, and the other facet is concerned with the allocation criterion for rewarding such outcomes. Consider Table 5.2, in which Borg and Staufenbiel (1993) coded 13 work value items in terms of seven facets. The facets and the structuples were taken from the literature on organizational behavior. Moorhead and Griffin (1989) argue, for example, that security in Maslow's sense interlocks with both Alderfer's relatedness and existence, but an item that is both Maslow-type security and Alderfer-type relatedness (item 10 in Table 5.2) is missing in the given sample of items.

TABLE 5.2. Work value items with various facet codings: H(erzberg) = {h = hygiene, m = motivators}; M(aslow) = {p = physiological, s = security, b = belongingness, r = recognition, a = self-actualization }; A(lderfer) = {e = existence, r = relations, g = growth}; E(lizur) = {i = instrumental-material, k = cognitive, a = affective-social}; R(osenberg) = {e = extrinsic, i = intrinsic, s = social}; L(evy-Guttman) = {i = independent of individual performance, g = depends on group performance, n = not performance dependent}; B(org-Elizur) = {1 = depends much on individual performance, 2 = depends more on individual performance than on system, 3 = depends both on individual performance and on system, 4 = depends on system only}.

Item	H	M	A	E	R	L	B	Work Value
1	m	a	g	k	i	g	3	Interesting work
2	m	a	g	k	i	g	3	Independence in work
3	m	a	g	k	i	g	3	Work that requires much responsibility
4	m	a	g	k	i	n	4	Job that is meaningful and sensible
5	m	r	g	k	e	i	1	Good chances for advancement
6	m	r	r	a	s	i	1	Job that is recognized and respected
7	h	b	r	a	s	n	4	Job where one can help others
8	h	b	r	a	s	n	4	Job useful for society
9	h	b	r	a	s	n	4	Job with much contact with other people
10	-	s	r	-	-	-	-	(No item of this type asked in study)
11	h	s	e	i	e	i	2	Secure position
12	h	s	e	i	e	i	1	High income
13	h	p	e	i	e	n	4	Job that leaves much spare time
14	h	p	e	i	e	n	4	Safe and healthy working conditions

Figure 5.5 shows a 2D MDS representation for the correlations of the 13 work value items assessed in a representative German sample. The radex partitioning is based on the facets "M(aslow)" (solid radial lines), "R(osenberg)" (dashed radial lines), and "L(evy-Guttman)" (concentric ellipses). It is easy to verify that the other facets also induce perfect and simple partitionings of this configuration. These partitionings are, moreover, quite similar: the respective regions turn out to be essentially congruent, with more or fewer subdivisions. Differences of the various wedge-like partitionings are primarily related to the outcome advancement, which is most ambiguous in terms of the need that it satisfies. Hence, one can conclude that all of these theories are structurally quite similar in terms of item intercorrelations. This suggests, for example, that Herzberg's motivation and hygiene factors correspond empirically to Elizur's cognitive and affective/instrumental values, respectively.

We note, moreover, that such similar partitionings of the MDS space into wedge-like regions—induced by different facets that are formally not equivalent—give rise to a partial order of the induced sectors. The interlocking of the Herzberg and the Maslow facets implies, for example, that the hygiene region contains the subregions "physiological", "security", and "belongingness", and the motivators' region contains the subregions "esteem" and "self-actualization". Hence, the subregions are forced into a certain neighborhood relation that would not be required without the hierarchical nesting. Similarly, the conceptual interlocking of the Maslow and the Alderfer facet requires "esteem" to fall between "self-actualization" and 'belongingness".

Elizur, Borg, Hunt, and Magyari-Beck (1991) report further studies on work values, conducted in different countries, which show essentially the same radex lawfulness. Note that this does not imply similarity of MDS configurations in the sense that these configurations can be brought, by admissible transformations, to a complete match, point by point (for such matchings; see Chapter 20). Rather, what is meant here is that several configurations (which do not even have to have the same number of points) exhibit the same law of formation: they can all be partitioned in essentially the same way (i.e., in the sense of a radex) by just one fixed coding of the items, thus showing similar *contiguity patterns* (Shye, 1981).

5.4 Partitioning MDS Spaces Using Facet Diagrams

Partitioning an MDS space is facilitated by using *facet diagrams*. Facet diagrams are simply subspaces—usually 2D projection planes—of the MDS space where the points are labeled by their structuples or, better, by their codings on just one facet (*structs*). This usually enables one to see the distribution of the points in terms of the particular typology articulated by each facet.

Consider an example that also explicates further aspects of facet theory (Galinat & Borg, 1987). In experimental investigations a number of properties of a situation have been shown, one by one, to have an effect on judgments of duration of time. The following mapping sentence shows four of these properties within a design meant to measure symbolic duration judgments, that is, duration judgments on hypothetical situations.

$$\text{Person } \{p\} \text{ believes that the } \overset{\underline{\text{positivity}}}{\left\{\begin{array}{l} p_1 = \text{pleasant} \\ p_2 = \text{neutral} \\ p_3 = \text{unpleasant} \end{array}\right\}} \text{ situation with}$$

$$\overset{\underline{\text{number}}}{\left\{\begin{array}{l} m_1 = \text{many} \\ m_2 = \text{few} \end{array}\right\}} \overset{\underline{\text{variability}}}{\left\{\begin{array}{l} v_2 = \text{monotonous} \\ v_1 = \text{variable} \end{array}\right\}} \text{ events that are}$$

$$\overset{\underline{\text{difficulty}}}{\left\{\begin{array}{l} s_1 = \text{difficult} \\ s_2 = \text{easy} \end{array}\right\}} \text{ to handle is felt as} \rightarrow \overset{\underline{\text{reaction}}}{\left\{\begin{array}{c} \text{very short in duration} \\ \text{to} \\ \text{very long in duration} \end{array}\right\}}.$$

The mapping sentence first shows a placeholder for the population of respondents $\{p\}$. In each particular way of reading the mapping sentence, one element of $\{p\}$ is picked and crossed with one particular combination of the elements of the *content facets*. The content facets distinguish among different situations by considering four properties of its events: "positivity of events", "number of events", "variability of events", and "difficulty to handle events". With the number of facet elements we have specified here—3 on "positivity", 2 on "number", 2 on "variability", and 2 on "difficulty"—we have $3 \cdot 2 \cdot 2 \cdot 2 = 24$ different situation types. For example, a situation with structuple (p_3, m_2, v_1, s_2) or, for short, 3212 is defined to be an unpleasant one, where few things are happening, with much variability, and no problems to cope with what is going on.

What we are interested in is how persons judge the duration of these 24 situation types. The mapping sentence identifies the characteristics of these situations in a relatively abstract way. For each type of situation, concrete examples must be constructed in order to have items that can be presented to respondents for assessment. The following story illustrates a concrete item for a situation of type $p_1 m_1 v_1 s_2$. "You are playing a simple card game with your children. It is quite easy for you to win this game because your kids are no serious opponents. The game requires you to exchange many different cards. The game is fun throughout the three minutes that it lasts." This description is supplemented by the question, "What do you think; how long would this card game seem to last? Would it seem longer or shorter than three minutes?"

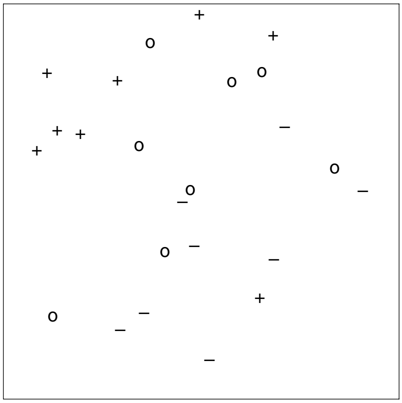

FIGURE 5.6. Facet diagram for duration judgments and facet "positivity".

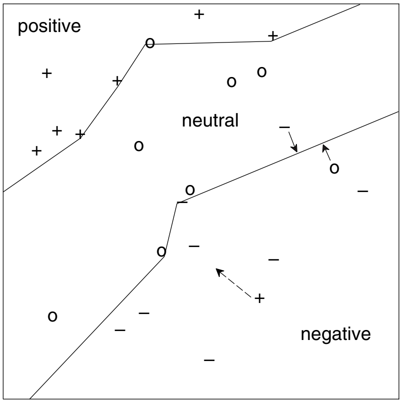

FIGURE 5.7. Facet diagram with axial partitioning.

A sample of persons rated this and 23 other hypothetical situations on a 7-point scale from "a lot shorter" (coded as 1) to "a lot longer." This bipolar scale, together with the question, "What do you think: how long ...?", is a concrete specification for the generic response range "very long ... very short in duration" in the above mapping sentence.

The intercorrelations of the 24 items are mapped into a 4D MDS space (with Stress = .13). Four dimensions are chosen because we assume that each facet can be completely crossed with any other. We now look at this space in terms of two projection planes. Figure 5.6 shows the plane spanned by the first two principal axes of the MDS configuration. Its points are labeled by the structs of each point on the facet "positivity". That is, points labeled as – in this facet diagram represent situations defined as p_3 = unpleasant. (Instead of –, one could also have chosen p_3, "unpleasant", "neg", "3", or any other symbolism, of course.) The facet diagram shows immediately that the points marked as +, o, and – are not distributed randomly. Rather, the plane can be partitioned into regions so that each region contains only or almost only points of one particular type. Figure 5.7 shows such a partitioning. It contains two minor errors: the two solid arrows indicate where these points "should" lie to be in the appropriate regions. Obviously, they are not far from the boundaries of these regions. There is also one further, and gross, error: a "positive" point located in the "negative" region. The dashed arrow attached to this point indicates the direction of required shifting.

Figure 5.8 represents an alternative partitioning that is error-free. This partitioning depends, however, very much on the position of the one point marked by an arrow. Thus, it may be less reliable in further replications. Moreover, the two partitionings imply different things. The concentric regions of Figure 5.8 predict that the duration ratings on unpleasant situa-

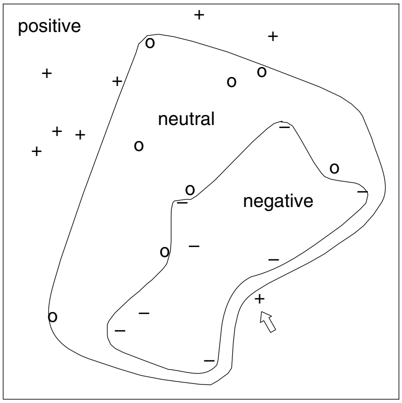

FIGURE 5.8. Facet diagram with modular partitioning.

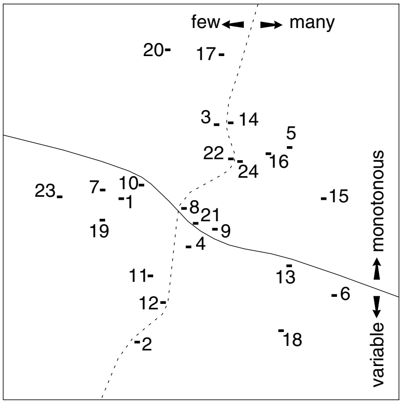

FIGURE 5.9. MDS projection plane of 24 duration situations, spanned by third and fourth principal components, partitioned by facets "variability" and "number".

tions should correlate higher among each other, on the average, than those for pleasant situations. The parallel regions of Figure 5.7 do not thus restrict the correlations. Nevertheless, both partitions are similar in splitting the plane into *ordered* regions, where the neutral region lies in between the positive and the negative regions. Hence, the regions are ordered as the facet "positivity" itself. Neither the spatial organization induced by the straight lines nor that induced by concentric circular lines would therefore have problems in accommodating a "positivity" facet, which distinguishes many more than just three levels. This is important because what we want, eventually, is not a theory about some particular sample of stimuli but one about the *universe* of such situation types. We thus see that the facet "positivity" is reflected in the structure of the duration ratings. The decision on which of the two partitionings is ultimately correct requires further data.

Figure 5.9 shows another plane of the 4D MDS space. It is spanned by principal axes 3 and 4 of the space and is therefore *orthogonal* to the plane in Figures 5.6–5.8. That is, each of its axes is perpendicular to both axes used in Figures 5.6–5.8. One recognizes from the respective facet diagrams (not shown here) that the configuration in this plane can be partitioned by the facet "number"—without error—and also by "variability"—with two errors.

The facet "difficulty" does not appear to show up in the MDS configuration; that is, the points representing easy and difficult situations, respectively, seem to be so scrambled that they can be discriminated only by very "irregular" partitionings. Such partitionings are, however, rarely useful. Note, though, that just looking at various orthogonal planes does not guarantee that one will detect existing regional patterns because such

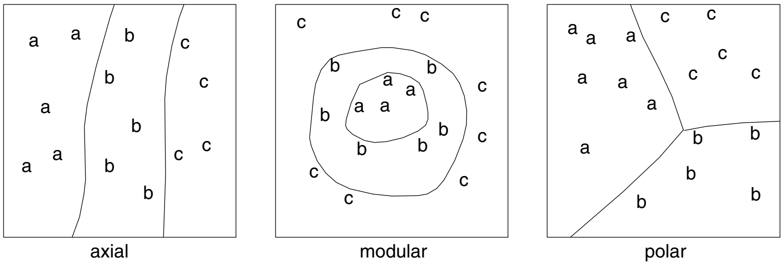

FIGURE 5.10. Three prototypical roles of facets in partitioning a facet diagram: axial (left panel), modular (center), and polar (right).

patterns may be positioned obliquely in space. This remains an unsolved problem that is particularly relevant in higher-dimensional spaces. In any case, using various spatial rotations and projections, we at least could not identify any simple regions related to the facet “difficulty” (Galinat & Borg, 1987).

MDS thus shows that the structure of the duration ratings can, in a way, be explained by three of the four facets of the design. This explanation is, moreover, compatible with considerations that extend beyond the sample of the given 24 concrete situations and that relate to their universe.

5.5 Prototypical Roles of Facets

With the partitionings shown in Figures 5.7 and 5.9, one arrives at an embryonic Cartesian coordinate system spanned by the three facets “positivity”, “number”, and “variability”. Another coordinate system is suggested if we accept the circular partitioning shown in Figure 5.8. In this case, we have some evidence for a *polar* coordinate system of these facets.

The coordination of the MDS configuration in these examples is not chosen arbitrarily. Rather, it relates naturally to content. We stress this point here because the data determine only the distances among the points, not any dimensions. Dimensions are either superimposed onto the distance geometry in order to be able to replace ruler-and-compass construction methods by computation, or they may result from projecting content onto the geometry, as we saw earlier.

The content facets often play one of three prototypical roles in this context. This is shown in the three panels of Figure 5.10. The panels exhibit schematic facet diagrams, whose points are labeled as a, b, and c. In the panel on the left-hand side, the space is partitioned in an *axial* way. The panel in the center shows a *modular* partitioning. The panel on the right-hand side shows a *polar* facet. An axial facet is one that corresponds to a dimension; that is, the partitioning lines cut the space into subspaces that look like parallel stripes of the plane (*axial simplex* of regions; see also Figure 5.7). A modular facet leads to a pattern that looks like a set of concentric bands (*radial simplex* of regions; see also Figure 5.8). Finally, a polar facet cuts the space, by rays emanating from a common origin, into sectors, similar to cutting a pie into pieces (*circumplex* of regions; see also Figure 5.3).

A number of particular combinations of facets that play such roles lead to structures that were given special names because they are encountered frequently in practice. For example, the combination of a polar facet and a modular facet in a plane, having a common center, constitutes a *radex* (see Figure 5.3). Adding an axial facet in the third dimension renders a *cylindrex*. Another interesting structure is a *multiplex*, a conjunction of at least two axial partitionings (see Figure 5.9). Special cases of the multiplex are called *duplex* (two axial facets), *triplex* (three axial facets), and so on. The multiplex corresponds to the usual (Cartesian) coordinate system ("dimensions") as a special case if the facets are (densely) ordered and the partitioning lines are straight, parallel, and orthogonal to each other.

There is also a variety of structures that are found less frequently in practice, for example, the *spherex* (polar facets in three-dimensional space) or the *conex* (similar to the cylindrex, but with radexes that shrink as one moves along its axial facet).

5.6 Criteria for Choosing Regions

Partitionings of geometric configurations that consist of only a few points are relatively easy to find. However, there is often so much leeway for choosing the partitioning lines that their exact shape remains quite indeterminate. More determinacy and greater falsifiability are brought in by increasing the number of items. Another principle for restricting the choice of partitioning lines is to think beyond the sample. In Figure 5.2, the partitioning lines were chosen, in part, by considering the universe of all intelligence items, a cylindrex.

Thinking beyond what was observed is always desirable, although it is, of course, impossible to say in general how this could be done. Most researchers typically are interested in generalizing their findings to the entire content universe, to additional populations, and over replications. The system of partitioning lines therefore should be *robust* in this respect, and not attend too much to the particular sample. Simple partitionings with relatively smooth cutting lines are typically more robust. But what is simple? Surely, a regionalization consisting of simply connected regions as in an axial or an angular system is simple, but so are the concentric bands of a circumplex. Hence, simple means, above all, that the partitioning is simple to characterize in terms of the roles of the facets that induce the regions. Naturally, if one admits greater irregularities (i.e., not requiring the lines to be so stiff locally), then the number of errors of classification can generally be reduced or even eliminated. However, such error reduction typically makes it more difficult to describe the structure and, as a consequence, makes it harder to express how the facets act on the MDS space. Moreover, irregular ad hoc partitionings also reduce the likelihood of finding similar structures in replications and in the universe of items. One thus faces a trade-off decision of the following kind. Should one use relatively simple partitionings at the expense of more errors? Or should one choose more irregular lines to avoid classification errors, and then leave it to the reader to simplify these patterns? Obviously, one has to decide what seems most appropriate in the given context.

Irregular lines cast doubts on the falsifiability of regional hypotheses. Partitionings become less likely to result from chance the more points they classify correctly, the more differentiated the system of facets is, the simpler the partitioning lines are, and the greater the stability of the pattern is over replications. For arbitrary structuples, one should not expect to find regional correspondences in the data. To see this, we simulate this case by randomly permuting the structuples in Table 5.1. Assume that this has led to the assignments 1 = GA, 2 = NI, 3 = GA, 4 = NA, 5 = GI, 6 = NA, 7 = GI, and 8 = GA. If we label the points in Figure 5.2 by these structuples, we find that the plane can be partitioned in a modular way by the facet {A, I}, but that the A-points are now in the center in between the I-points. That does not correspond to the structure of the content universe, the cylindrex, which was replicated in hundreds of data sets (Guttman & Levy, 1991). The second facet, {G, N}, leads to a partitioning line that winds itself snake-like through the circular MDS configuration. It thus shows that separating the G- from the N-points with a reasonably regular line is only possible because we have so few points. It can hardly be expected that such an artificial partitioning can be replicated in other and richer data sets.

In addition to these formal criteria, one must request that the pattern of regions also ultimately makes sense. Yet, irregular lines are already difficult to describe as such and, as a consequence, complicate the search for explaining the way in which the regions are related to the facets. Moreover, in the given case, the radial order of inference, application, and learning is not only replicable, but also seems to point to an ordered facet "complexity", where inference is the most complex task (see above). If application items, then, come to lie in the radex center, such further search for substantive meaning is thwarted.

To avoid seemingly arbitrary partitionings or to aid in partitioning MDS spaces, Shye (1991) proposed a computerized method for partitioning facet diagrams in three ways: (1) in an axial way, by parallel and straight lines; (2) in a modular way, by concentric circles; and (3) in a polar way, by rays emanating from a common origin. The program yields graphical displays of three optimal partitionings, and measurements of the goodness of these partitionings by providing a *facet separation index* based on the sum of distances of the "deviant" points from their respective regions and normalized by the separability that can be expected for random data (Borg & Shye, 1995). Using this procedure suggests, for example, that a concentric-circles partitioning is best in terms of separability for the facet E = {i = instrumental-material, k = cognitive, a = affective-social} for the configuration in Figure 5.5. This finding conflicts with our previous decision to use polar partitioning for the very similar facet suggested by Rosenberg. On closer inspection, one notes, however, that it hinges on the location of one point, that is "good chances for advancement." This work value was categorized by Elizur as cognitive, but for a representative sample it may be better categorized as instrumental-material, because higher pay, more job security, and better working conditions may be more what most people have in mind when they assess the importance of advancement. Another criterion that speaks against the concentric-circles partitioning is that it induces ordered regions. The concentric circles that lead to the best separability index for facet E with respect to the given MDS configuration place the affective region in between the instrumental region and the cognitive region. Hence, the regions are ordered in this partitioning, while the facet only makes nominal distinctions, and no rationale for this order seems obvious a posteriori, except that affective values may be more highly intercorrelated than cognitive or instrumental values, in general. Naturally, such content considerations, as well as generalizability and replicability, must be considered in addition to formal separability measures for a given sample representation.

5.7 Regions and Theory Construction

Definitions and data are intimately linked through correspondence hypotheses not only at a particular point in time, but they are also related to each other over time in a "partnership" (Guttman, 1991) of mutual feedback. The definitions serve to select and structure the observations. The data then lead to modifications, refinements, extensions, and generalizations in the definitional framework. There is no natural beginning of this partnership between data and definitions. Hence, a correspondence between data and definitions can also be established a posteriori. That is, one may recognize certain groupings or clusters of the points, and then think about a rationale afterwards to formulate new hypotheses. When the definitional framework is complex, one typically does not predict a full-fledged regional system (such as a cylindrex) unless past experience leads one to expect such a system. Rather, one uses a more modest strategy with exploratory characteristics, and simply tries to partition the space, facet by facet, with minimum error and simple partitioning lines. Even more liberal and exploratory is the attempt to identify space partitions according to new content facets that were not conceived in advance. The stability of such partitions is then tested in replications.

Replicating a regional correspondence, and thereby establishing an empirical law, is not sufficient for science. Researchers typically also want to understand the law. Why, for example, are work values organized in a radex? An answer to this question can be derived, in part, from reasoning in Schwarz and Bilsky (1987). These authors studied general values. One of the facets they used was "motivational domain" = {achievement, self-direction, security, enjoyment, ...}. These distinctions were considered nominal ones, but there was an additional notion of substantive *opposition*. Four such oppositions were discussed, for example, achievement vs. security: "To strive for success by using one's skills usually entails both causing some change in the social or physical environment and taking some risks that may be personally or socially unsettling. This contradicts the concern for preserving the status quo and for remaining psychologically and physically secure that is inherent in placing high priority on security values" (p. 554). Hence, the region of achievement values was predicted to lie opposite the security region. If we use this kind of reasoning post hoc on the work value radex of Figure 5.5, we could explain the opposite position of the sectors v and a (in Maslow's sense) by a certain notion of "contrast" of striving for self-actualization and for recognition, respectively. This notion of contrast is derived from a basic facet analysis of action systems (Shye, 1985). The same facet analysis also explains the neighborhood of regions like recognition and security, for example.

To predict regional patterns requires one to clarify the expected roles of the facets in the definitional framework. This involves, first of all, classifying the scale level of each facet. For ordered facets, one predicts a regional structure whose regions are also ordered so that the statement that some region R comes "before" another region R′ has meaning. The order of the regions should correspond to the order specified for the elements of the corresponding facet. For qualitative facets, any kind of simple partitionability of the point configuration into regions is interesting. The distinction of facets into *qualitative* and *ordinal* ones represents a "role assignment" (Velleman & Wilkinson, 1994) that is "not governed by something inherent in the data, but by interrelations between the data and some substantive problem" (Guttman, 1971, p. 339), that is, by certain correspondence hypotheses linking the observations and the definitional system. Hence, if one can see a conceptual order among the facet's elements and hypothesize that this order is mirrored in the observations collected on corresponding items, then the facet "is" ordered—for testing the hypothesis. Scale level thus remains context-related.

Consider as an example the facet "color" = {red, yellow, green, blue, purple}. One would be tempted to say, at first, that this "is" a nominal facet. Yet, with respect to similarity judgments on colors, "color" has been shown to be ordered empirically in a circular way (see Chapter 4). Furthermore, with respect to the physical wavelength of colors, "color" is linearly ordered.

5.8 Regions, Clusters, and Factors

As is often true with concepts used in FT relative to similar ones in data analysis, the FT notion is more general. An important example is that regions include clusters as a special case. Lingoes (1981) proposes a faceted way to distinguish among different types of regions. He suggests that a cluster is a particular region whose points are all closer to each other than to any point in some other region. This makes the points in a cluster look relatively densely packed, with "empty" space around the cluster. For regions, such a requirement generally is not relevant. All they require is a rule that allows one to decide whether a point lies within or outside the region. The points 5 and 6 in Figure 5.2 are in different regions, but complete linkage clustering (a common type of cluster analysis), for example, puts them into one cluster together with point 4, and assigns points 7 and 8 to another cluster. For regions, the distance of two points—on which clustering is based—does not matter. Indeed, two points can be very close and still be in different regions. Conversely, two points may be far apart and still belong to the same region. As an analogy, consider Detroit (Michigan) and Windsor (Ontario). These cities are much closer than Detroit and Los Angeles, for example, but Detroit and Los Angeles are both in the same country, whereas Detroit and Windsor are not. In regions, all that counts is discriminability. Moreover, clusters are usually identified on purely formal criteria, whereas regions are always based on substantive codings of the represented objects. Guttman (1977) commented therefore as follows: "... theories about non-physical spaces ... generally call for continuity, with no 'vacuum' or no clear separation between regions... The varied data analysis techniques going under the name of 'cluster analysis' generally have no rationale as to why systematic 'clusters' should be expected at all... The term 'cluster' is often used when 'region' is more appropriate, requiring an outside criterion for delineation of boundaries" (p. 105).

Factors from factor analyses are not directly related to regions or to clusters. However, it is often asked in practice what one would have found if one had analyzed a correlation matrix by factor analysis rather than by MDS. Factor analysis, like cluster analysis, is a procedure that is substantively "blind" (Guttman, 1977) or that, if used in a confirmatory way, forces a preconceived formal structure onto the data representation, namely "factors". The factors are (rectilinear) dimensions that are run through point clusters, usually under the additional constraint of mutual orthogonality.

For Table 5.1, a factor analysis yields three factors with eigenvalues greater than 1. After varimax rotation, one finds that these factors correspond to three clusters in Figure 5.1, {1,2,3}, {4,5,6}, and {6,7,8}. Hence, in a way, the factors correspond to a polar partitioning of the MDS configuration in the given case, with three factors or "regions" in a 2D MDS space. With positive correlation matrices, this finding is rather typical; that is, one can expect $m+1$ factor-induced regions in an m-dimensional MDS space. The reason for this is that positive correlations are conceived of in factor analysis as a vector bundle that lies in the positive hyperoctant of the Cartesian representation space, whereas MDS—which does not fix the origin of the space—looks only at the surface that contains the vector endpoints. Thus, Figure 5.1 roughly shows the surface of a section of the sphere whose origin lies somewhere in the center of the points but behind (or above) the plane (Guttman, 1982). The factors then correspond to a tripod fixed to the origin and rotated such that its axes lie as close as possible to the points. Hence, one notes that the location of this dimension system is highly dependent on the distribution of the points in space, whereas this is irrelevant for regions, although, of course, a very uneven distribution of the points in space will influence the MDS solution through the Stress criterion.

5.9 Exercises

Exercise 5.1 Consider the multitrait-multimethod matrix below (Bagozzi, 1993). It shows the correlations among nine items. The items assess the traits global self-esteem, social self-esteem, and need for order. Each trait is measured by three methods: true–false, multipoint, and simple self-rating scales.

Item	No.	1	2	3	4	5	6	7	8	9
T_1M_1	1	(.83)								
T_2M_1	2	.58	(.85)							
T_3M_1	3	.17	.14	(.74)						
T_1M_2	4	.75	.45	.23	(.93)					
T_2M_2	5	.72	.74	.16	.65	(.91)				
T_2M_2	6	.09	.06	.68	.25	.08	(.85)			
T_1M_3	7	.58	.53	.14	.62	.68	.09	(.63)		
T_2M_3	8	.47	.74	.10	.40	.69	.07	.58	(.74)	
T_3M_3	9	.22	.18	.63	.34	.22	.56	.30	.23	(.82)

(a) Do an MDS of this data matrix and check the configuration for possible correspondences to the trait and the method facet, respectively. Try both 2D and 3D solutions.

(b) What can you conclude about the relative weight of trait and method in these data?

(c) Characterize the roles of facets T and M in this MDS configuration.

(d) Compare the roles of facets T and M to the roles that T and M play in Exercise 1.6.

Exercise 5.2 Consider the data matrix below based on a representative survey in the U.S.A. It shows the intercorrelations of items asking about satisfaction with different aspects of one's life. According to Levy (1976), one can classify these items by the following mapping sentence. The extent of satisfaction of respondent x with the $\{a_1$ = state of, a_2 = resources for$\}$ his or her activities in area of life $\{b_1$ = education, b_2 = economy, b_3 = residence, b_4 = spare time, b_5 = family, b_6 = health, b_7 = work, b_8 = general$\}$ $\rightarrow$ {very positive ... very negative} satisfaction with life.

Item	A	B	1	2	3	4	5	6	7	8	9	10	11	12	13	14
1 City as place to live	2	3														
2 Neighborhood	2	3	54													
3 Housing	2	3	44	49												
4 Life in the U.S.	2	3	33	28	29											
5 Amount of educat.	2	1	19	18	23	12										
6 Useful education	2	1	14	14	19	15	54									
7 Job	1	7	22	21	26	23	25	24								
8 Spare time	1	4	22	19	27	23	26	23	33							
9 Health	2	9	05	00	06	06	18	17	13	21						
10 Standard of living	1	2	33	32	45	24	32	24	35	37	17					
11 Savings, investmt.	2	2	25	23	29	19	28	20	27	32	17	59				
12 Friendships	2	4	24	19	23	21	16	17	25	40	09	25	24			
13 Marriage	2	5	14	13	21	13	09	12	25	30	12	25	23	21		
14 Family life	1	5	24	19	23	21	18	18	27	40	14	32	25	31	48	
15 Life in general	1	8	28	23	30	24	28	24	34	50	26	45	36	32	38	50

(a) According to Levy facets A and B establish a radex in a 2D MDS representation of these data. Verify.

(b) Characterize the roles of facets A and B in the MDS space.

(c) What item lies at the origin of the radex? Can you give a substantive explanation of why this makes sense?

(d) Items that lie more at the center of the radex are more similar to each other. What does that mean in this particular context?

Exercise 5.3 Consider the data matrix below. It shows the correlations for 12 intelligence tasks from the Wechsler test. The coefficients below the main diagonal are based on 2200 U.S. children; the coefficients above the main diagonal come from 1097 Israeli children. Following Guttman and Levy (1991), the tasks can be described by the following mapping sentence. The correctness of the response of testee x to a task that requires {I = inference, A = application, L = learning} of an objective *rule* through {o = oral, m =

manual manipulation, p = paper and pencil} *expression* $\rightarrow$ {high ... low} correctness.

Item	Rule	Exp	1	2	3	4	5	6	7	8	9	10	11	12
1 Information	A	o		51	52	58	46	36	40	38	42	34	31	30
2 Similarities	I	o	62		42	58	49	31	36	41	41	35	29	25
3 Arithmetic	A	o	54	47		44	36	43	34	33	44	33	33	32
4 Vocabulary	A	o	69	67	52		60	35	41	44	41	37	31	27
5 Comprehension	I	o	55	59	44	66		24	38	40	38	36	30	30
6 Digit span	L	o	36	34	45	38	26		28	28	32	23	29	26
7 Picture completion	A	o	40	46	34	43	41	21		45	47	45	25	31
8 Picture arrangement	A	m	42	41	30	44	40	22	40		45	48	28	35
9 Block design	A	m	48	50	46	48	44	31	52	46		57	32	39
10 Object assembly	A	m	40	41	29	39	37	21	48	42	60		27	40
11 Coding	L	p	28	28	32	32	26	29	19	25	33	24		23
12 Mazes	L	p	27	28	27	27	29	22	34	32	44	37	21	

(a) Do an MDS analysis of both the U.S. and the Israeli correlation matrices.

(b) Check whether the facets *rule* and *expression* allow you to structure ("explain") the MDS configurations.

(c) Characterize the roles these facets play in the MDS spaces.

(d) Which tasks are more central ones in terms of the spatial regions? Discuss in substantive terms what it means that "the closer an intelligence item is to being a 'rule inference', the weaker its affinity is to a single kind of material" (Shye, Elizur, & Hoffman, 1994)[p. 112]. ("Material" here corresponds to what Guttman calls "expression".)

Exercise 5.4 Consider the MDS configuration in Figure 5.5. Its interpretation is based on regions induced by some of the facets exhibited in Table 5.2. A special case of a region is a cluster. Clusters may emerge "out of substance" when one partitions an MDS space by facets defined for the entities represented by the points. However, clusters sometimes are also used in the MDS context in a purely exploratory way to help interpret MDS solutions. For that purpose, the proximities are subjected to a hierarchical cluster analysis, and the emerging cluster hierarchy is superimposed onto the MDS plane by expressing each cluster as a convex hull around the points that belong to the cluster. With hierarchical clusters, this often leads to *families* of such hulls that look like altitude or contour lines on a geographic map. We now use this approach on the data on which Figure 5.5 is based. These data are shown in the table below.

No.	Work Value	1	2	3	4	5	6	7	8	9	10	11	12	13
1	Interesting													
2	Independence	.44												
3	Responsibility	.61	.58											
4	Meaningful work	.49	.48	.53										
5	Advancement	.32	.44	.33	.39									
6	Recognition	.39	.34	.41	.47	.38								
7	Help others	.38	.35	.41	.45	.27	.65							
8	Contribute society	.36	.29	.44	.43	.16	.49	.64						
9	Contact people	.21	.10	.22	.21	.16	.29	.35	.45					
10	Security	.28	.18	.30	.39	.15	.36	.37	.49	.61				
11	Income	.37	.32	.36	.46	.21	.33	.45	.45	.43	.68			
12	Spare time	.32	.29	.35	.34	.23	.56	.49	.44	.40	.47	.49		
13	Working cond.	.50	.37	.39	.40	.30	.45	.44	.35	.26	.37	.37	.60	

(a) Do a hierarchical cluster analysis on the work values correlations. Plot the resulting clusters as nested "altitude" lines onto the MDS plane for the same data.

(b) Compare the cluster structure to the regions in Figure 5.5. Discuss where they agree and where they differ.

(c) Cluster analysis is sometimes used to check whether the clusters that one sees in an MDS solution are but scaling artifacts. Green & Rao write: "As a supplementary step, the ... data ... were submitted to ... [a] clustering program ... the program was employed to determine how well the low-dimensional scaling solutions preserved the original relationships in the input data" (Green & Rao, 1972, p. 33). Discuss what they mean by that statement.

(d) Superimpose hierarchical clusters onto the similarity of nations data in Table 1.3.

(e) Test out different clustering criteria (in particular, single linkage and average linkage) and check how they differ in clustering the points of Figure 1.5. Discuss why they differ.

Exercise 5.5 Facets are often superimposed by the substantive researcher on a theoretical basis. The facets, then, are typically not obtrusive ones, and many alternative facetizations are possible using different theories. Yet, facets can also be obtrusive features of the entities. That is true, for example, for the items in factorial surveys ("vignettes") or for stimuli within a factorial design. In these cases, the objects possess a certain facet profile by construction. It is also true for the following matrix which shows rank-order correlations of favorite leisure activities for groups defined by gender, race, and self-defined social class (Shinew, Floyd, McGuire, & Noe, 1995).

No.	Group	1	2	3	4	5	6	7	8
1	Lower-class black women	–							
2	Middle-class black women	.71	–						
3	Lower-class black men	.54	.54	–					
4	Middle-class black men	.35	.45	.61	–				
5	Lower-class white women	.23	.52	.17	.55	–			
6	Middle-class white women	.29	.66	.20	.52	.77	–		
7	Lower-class white men	.20	.33	.51	.87	.54	.41	–	
8	Middle-class white men	.11	.07	.25	.81	.51	.26	.26	–

(a) Represent these data in an MDS plane.

(b) Partition the space by the facets gender, race, and class, respectively.

(c) Discuss the resulting regions. Which facets show up in simple regions; which facets do not? What do you conclude about the leisure activities of these groups?